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$$\tan \frac{1}{2}(A - B) = \frac{AJ}{BJ}; \quad \tan \frac{1}{2}(A + B) = \frac{VJ}{BJ};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{AJ}{VJ} = \frac{AK}{BV} = \frac{a - b}{a + b};$$

or

$$\tan \frac{1}{2}(A - B) = \frac{AJ}{BJ} = \frac{AK \sin \frac{1}{2}(A + B)}{BV \cos \frac{1}{2}(A + B)} = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B),$$

(6) Circumscribe a circle about ABC .

Draw CP bisecting the angle C . Make $CR = CA = b$. Then $BR = a - b$.

$$\angle CPS = \angle CAR = \frac{1}{2}(A + B),$$

$$\angle BPS = \angle BAR = \frac{1}{2}(A - B),$$

$$\angle BQS = \angle RQS = \angle BCP = \frac{C}{2},$$

or arc $BP =$ arc PA . Hence

$$\angle RSQ = 90^\circ$$

and

$$BS = SR = \frac{1}{2}(a - b),$$

$$CS = \frac{1}{2}(a + b).$$

Then we have

$$\tan \frac{1}{2}(A - B) = \frac{BS}{PS}; \quad \tan \frac{1}{2}(A + B) = \frac{SC}{PS};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{BS}{CS} = \frac{a - b}{a + b}.$$

In addition to the proofs here given I find distinct proofs in R. E. Moritz, *Elements of Plane Trigonometry*, p. 131; Hall and Frink, *Trigonometry*, p. 54; F. Durell, *Plane Trigonometry*, p. 108; in W. F. Cheney, 1920, 53-54; and in works by the following authors listed 1920, 53: E. J. Wilczynski, Young and Morgan, and E. W. Hobson.¹

II. NOTE ON THE LAW OF THE MEAN.

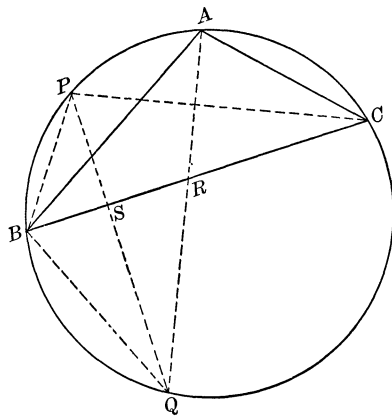
By H. H. DOWNING, University of Kentucky.

With certain well-known restrictions on $f(x)$ and its derivative, $f'(x)$, the law of the mean states that

$$\frac{f(x + h) - f(x)}{h} = f'(x + \theta h), \quad (1)$$

where θ is a positive proper fraction.

¹ Yet another proof is given by G. Lindborg in *Tidskrift för Elementär Matematik, Fysik och Kemi*, vol. 3, May, 1920, p. 201.—EDITOR.



It is desired in this paper to find a relation between θ and h for certain functions. To do this all restrictions on θ and h are removed, allowing them to have any values which are permissible as determined by the functional relation between them. The following cases will be considered:

Case 1. Consider the function $f(x) = ax + b$, $a \neq 0$.

Here $f'(x) = a = f'(x + \theta h)$. $f(x + h) = ax + ah + b$. These values substituted in (1) give the identity $a = a$, from which it is seen that no relation exists between θ and h . This is verified by the geometric interpretation of the tangent to the line whose equation is $y = ax + b$.

Case 2. Consider the function $f(x) = ax^2 + bx + c$, $a \neq 0$.

Obtaining $f'(x + \theta h)$, $f(x + h)$, and substituting in (1) as before it is found that $ah = 2a\theta h$. From this either $h = 0$ or $\theta = 1/2$. The first, $h = 0$, being trivial, geometrically, will not be considered further. The second, $\theta = 1/2$, clearly means that the ordinate of the point of tangency of the tangent which is parallel to the chord joining the points on the parabola $y = ax^2 + bx + c$ with abscissas x and $x + h$ is midway between the ordinates of these points.

Case 3. Next consider $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$.

Substitution in (1) gives $ah^2 + bh + 3ahx = 6a\theta hx + 3a\theta^2 h^2 + 2b\theta h$. To simplify this equation put $\theta h = \phi$, and the equation then reduces to

$$\frac{\left(h + \frac{3ax + b}{2a}\right)^2}{\left(\frac{3ax + b}{2a\sqrt{3}}\right)^2} - \frac{\left(\phi + \frac{3ax + b}{3a}\right)^2}{\left(\frac{3ax + b}{6a}\right)^2} = -1.$$

This is seen to represent a hyperbola with its principal axis parallel to the ϕ -axis. In this figure θ is the slope of the line from the origin to the point (h, ϕ) .

The following results are obtained:

The center is the point $(h, \phi) \equiv \left(-\frac{3ax + b}{2a}, -\frac{3ax + b}{3a}\right)$.

The semi-axes are: transverse $= \frac{3ax + b}{6a}$, conjugate $= \frac{3ax + b}{2a\sqrt{3}}$.

The eccentricity $= 2$.

The foci are: $F \equiv \left(-\frac{3ax + b}{2a}, -\frac{2(3ax + b)}{3a}\right)$ and $F' \equiv \left(-\frac{3ax + b}{2a}, 0\right)$.

The asymptotes are the lines $\phi = \frac{\sqrt{3}}{3}h + \frac{3ax + b}{6a}(\sqrt{3} - 2)$ whose inclination is 30° , and $\phi = -\frac{\sqrt{3}}{3}h - \frac{3ax + b}{6a}(\sqrt{3} + 2)$ whose inclination is 150° .

The vertices are the points

$$V \equiv \left(-\frac{3ax + b}{2a}, -\frac{3ax + b}{2a}\right) \quad \text{and} \quad V' \equiv \left(-\frac{3ax + b}{2a}, -\frac{3ax + b}{6a}\right).$$

It is seen that for F' , V' , C (the center), V and F , we have

$$\phi = 0, \quad \phi = \frac{1}{3}h, \quad \phi = \frac{2}{3}h, \quad \phi = h, \quad \phi = \frac{4}{3}h.$$

If x increases these points move towards the left along straight lines through the origin, and the hyperbolas corresponding to two values of x are homothetic with respect to the origin.

The interesting point about the equations is the nature of the coefficients of h .

It is further interesting to notice those points on the hyperbola which may be used when the conditions of the law of the mean are satisfied: that is, when θ is positive and less than one, or when ϕ is numerically less than h , and has the same sign. Also, notice may be made of these relations when the point on the cubic whose abscissa is x , is to the left of the inflection point, coincides with the inflection point, or is to the right of it.

RECENT PUBLICATIONS.

REVIEWS.

The Early Mathematical Manuscripts of Leibniz translated from the Latin texts published by Carl Immanuel Gerhardt with critical and historical notes. By J. M. CHILD, Chicago-London, The Open Court Publishing Co., 1920. Price \$2.00.

American readers will be glad to possess in English translation the manuscripts of Leibniz containing his early work in the creation of the calculus that were preserved in the Royal Library in Hanover and were first published by C. I. Gerhardt in the years 1846-1855. We have here seventeen manuscripts bearing dates which fall within the time 1673-1677 and also two manuscripts of later date which give two accounts, one brief, the other more extended, of the origin of the calculus—both from the pen of Leibniz himself. Child breaks the chronological order by printing the last two documents first, thus laying stress on the parts which after all are really less important; for it is the papers of 1673-1677 that contain the cardinal points of the record. The book begins with the controversy on the priority of discovery which, by rights, should have been postponed to the end. Child translates also Gerhardt's essay on "Leibniz in London," which includes three manuscripts of Leibniz, and a second essay by Gerhardt, on "Leibniz and Pascal." Child adds profuse historical notes and arguments.

As a rule, the style is clear, but we mention a few exceptions. On page 8, line 23, a sentence is quoted, but it does not clearly appear which of three men mentioned is the author of it; only by inference from what follows later can the reader place the author with certainty. On the same page, line 26, Child quotes from "the memoir referred to," when as yet, no memoir has been mentioned; the authorship of the quotation at the bottom of page 8 is not revealed in the text.

In some instances Child is guilty of inconsistencies and errors. On page 9 he gives 1708, instead of 1704, as the date of the first complete publication of Newton's *Tractatus de quadratura curvarum*. On page 13, line 38, the date of the second edition of Barrow's *Lectiones Opticæ et Geometricæ* is given as 1874; of course, it should be 1674. On page 6 Child declares that Fatio concludes that